Calculate trigonometric functions using the unit circle

A tutorial

Pooya Taherkhani

19 July 2025

Contents

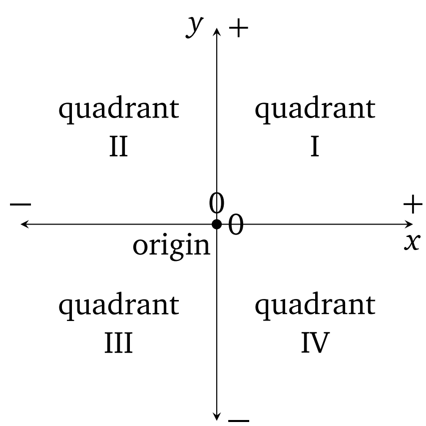
Reading reference:

You may find a related discussion of trigonometric functions in chapter 5 of OpenStax Precalculus textbook:

https://openstax.org/books/precalculus-2e/pages/5-introduction-to-trigonometric-functions

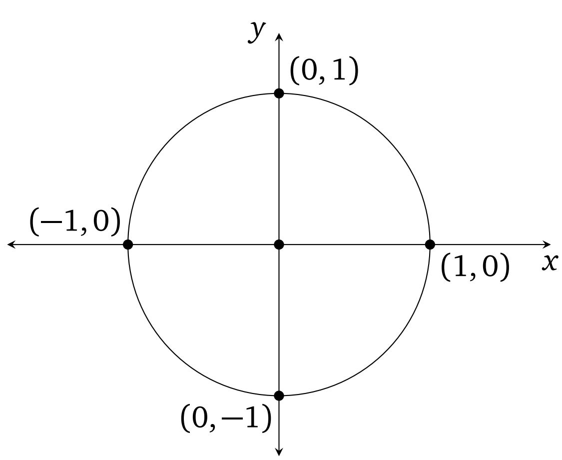
1 Cartesian coordinates

- The *Cartesian coordinate system*, also known as the *rectangular coordinate system*, has two perpendicular axes intersecting at the *origin*.
- The horizontal axis in the system is usually labeled x and the vertical axis is labeled y.
- Cartesian coordinates are named after *René Descartes*.
- All real numbers can be mapped to either axes, x or y, with 0 being mapped to the origin, and positive numbers to the *right* half of the x axis or the *upper* half of the y axis.
- The Cartesian coordinate system divides the plane into four quadrants—see the diagram.



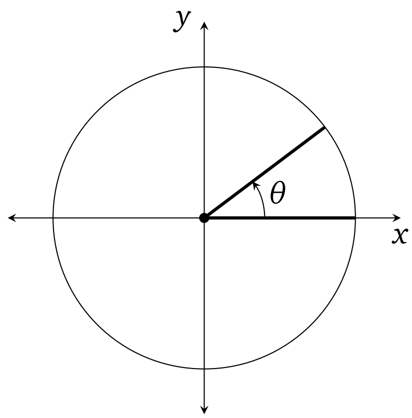
1.1 Unit circle

- The *unit circle* is a circle with a radius of 1 centered at the origin of the Cartesian coordinate system.



2 Angles

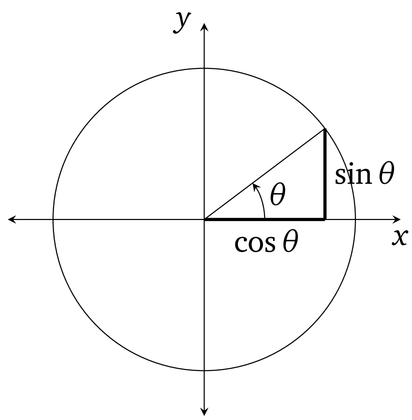
- Consider the angle θ whose vertex conincides with the origin.



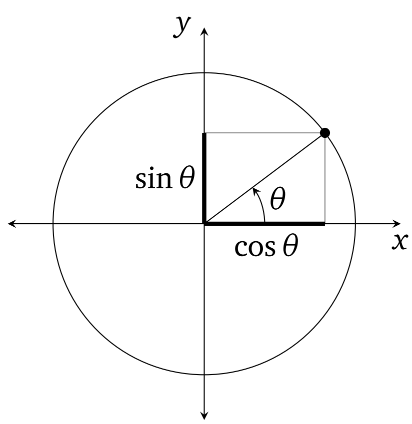
- Let's fix one side of the angle θ on the positive direction of the x axis.
  - This side is called the *initial side*.
- The other side of the angle can rotate and land anywhere around the circle.
  - We call this side the *terminal side*.
- Let's set *counterclockwise* as the positive direction for the angle θ.
- If θ is the angle that rotates one full cycle around the circle, by convention we say that $\theta = 360°$, which is read *theta equals 360 degrees*.
- Degree is just a unit of measrement of angles, developed by ancient astronomers who had developed 360-day calendars!
- We will later discuss another unit of measurement for angles, called *radians*.

3 sin and cos functions

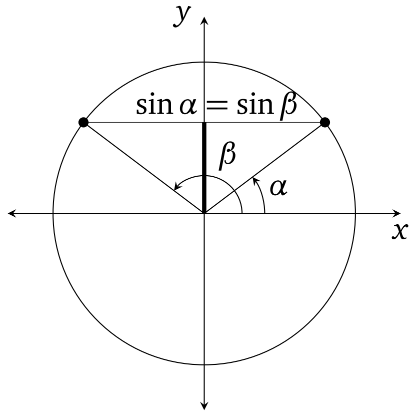
- Consider a right trinagle whose hopytenuse is the terminal side of the angle θ in the unit circle, and whose sides are parallel to the x and y axes.
- *Sine* and *cosine* of θ are by definition the (signed) length of the sides of the above-defined right triangle.
  - $\cos\theta$ is the length of the side parallel to the x axis.
  - and $\sin\theta$ is the length of the side parallel to the y axis.



- Note that the definition above is equivalent to the definition of sine or cosine being respectively the opposite side or the adjacent side over the hypotenuse, considering that the hypotenuse in the setup above is of length 1.
- The side of the right traingle corresponding to $\sin\theta$ can be projected onto the y axis, as shown below.



- So given the terminal side of the angle θ, we can calculate $\cos\theta$ by projecting the terminal side on the x axis, and $\sin\theta$ by projecting on the y axis.
- Note that the value of sin and cos functions can be negative, zero, or positive.
- Note that since the unit circle is symmetric, different angles with different terminal sides may have the same sine *or* cosine value!



3.1 Exercise

Using the unit circle determine the sine and cosine of the following angles:

1. $0°$
2. $90°$
3. $180°$
4. $270°$
5. $360°$

Solutions:

1. $\sin 0° = 0$, $\cos 0° = 1$
2. $\sin 90° = 1$, $\cos 90° = 0$
3. $\sin 180° = 0$, $\cos 180° = -1$
4. $\sin 270° = -1$, $\cos 270° = 0$
5. $\sin 360° = 0$, $\cos 360° = 1$

3.2 Exercise

Using the unit circle, determine the sign of the sine and cosine of the following angles.

6. The terminal side of θ falls in the first quadrant.
7. The terminal side is in the second quadrant.
8. The terminal side is in the third quadrant.
9. The terminal side is in the fourth quadrant.

Solutions:

6. $\sin\theta > 0$, $\cos\theta > 0$
7. $\sin\theta > 0$, $\cos\theta < 0$

8. $\sin\theta < 0$, $\cos\theta < 0$
9. $\sin\theta < 0$, $\cos\theta > 0$

3.3 Exercise

Use the unit circle to answer the following questions.

10. What is the *maximum* possible value of the *sine* function? At what value of θ this maximum occurs?
11. What is the *minimum* possible value of the *sine* function? At what value of θ this minimum occurs?
12. What is the range of the *sine* function?

    (What is the range of the values of the *sine* function for any value of θ?)
13. What is the *maximum* possible value of the *cosine* function? At what value of θ this maximum occurs?
14. What is the *minimum* possible value of the *cosine* function? At what value of θ this minimum occurs?
15. What is the range of the *cosine* function?

    (What is the range of the values of the *cosine* function for any value of θ?)

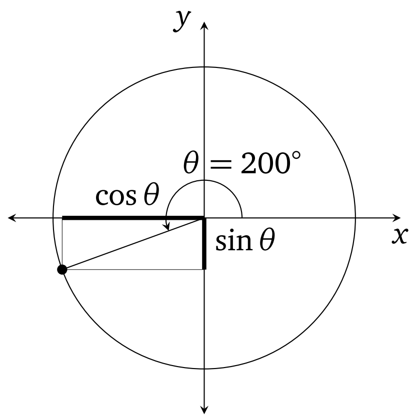
Solutions:

10. 1, 90° because $\sin 90^\circ = 1$
11. -1, 270° because $\sin 270^\circ = -1$
12. $-1 \leqslant \sin\theta \leqslant 1$ for any θ
13. 1, 0° because $\cos 0^\circ = 1$
14. -1, 180° because $\sin 180^\circ = -1$
15. $-1 \leqslant \cos\theta \leqslant 1$ for any θ

3.4 Exercise

16. Draw the unit circle for $\theta = 200^\circ$ and determine the line segments on the x and y axes corresponding to $\sin\theta$ and $\cos\theta$.

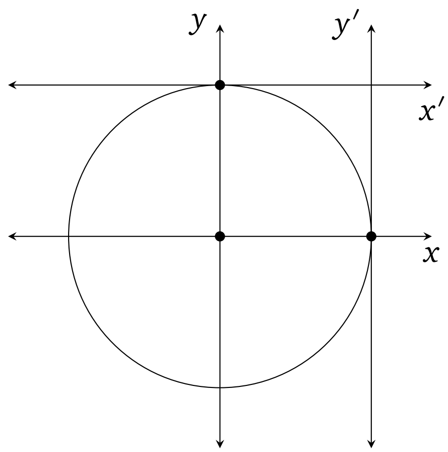
Solution:



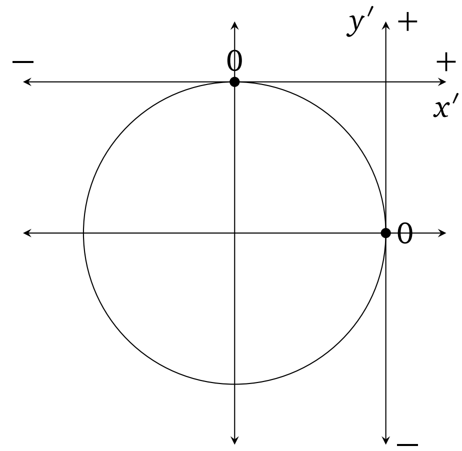
Note that $\sin 200^\circ$ and $\cos 200^\circ$ are both *negative*.

4 tan **and** cot **functions**

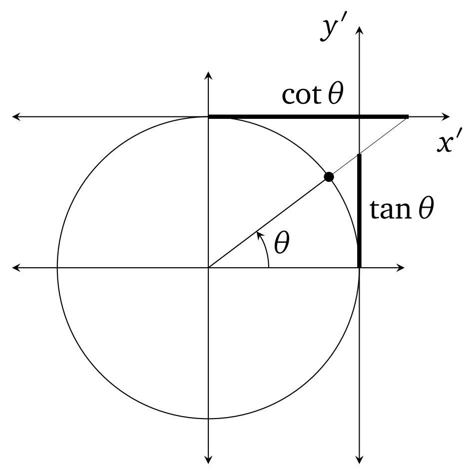
- Now we introduce a second pair of axes, namely x' and y', which will help us to calculate *tangent* and *cotangent* of an angle.



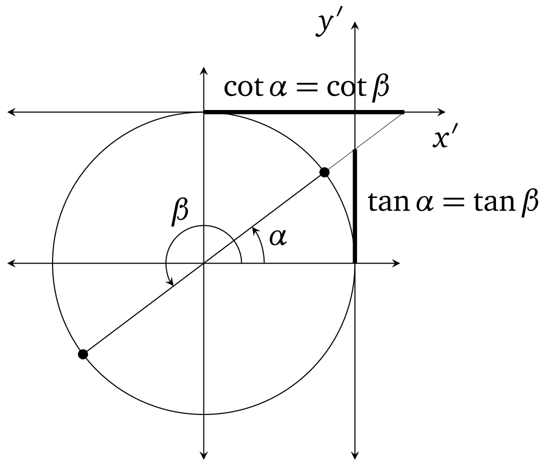
- The x' axis is produced by moving the x axis upward until it becomes tangent to the unit circle.
- Similarly, the y' axis is produced by moving the y axis to the right until it becomes tangent to the unit circle.
- In the figure below notice the location of the *zeros* on the x' and the y' axes.



- We calculate cotangent on the x' axis, and tangent on the y' axis, by extending the terminal side of the angle θ.
  - The intersecting point with the x' axis yields $\cot\theta$.
  - And the intersecting point with the y' axis yields $\tan\theta$.



- Note that the value of tangent and cotangent functions can be negative, zero, or positive.
- Note that because of the symmetry of the unit circle, angles with different terminal sides may have the same tangent *and* cotangent values!



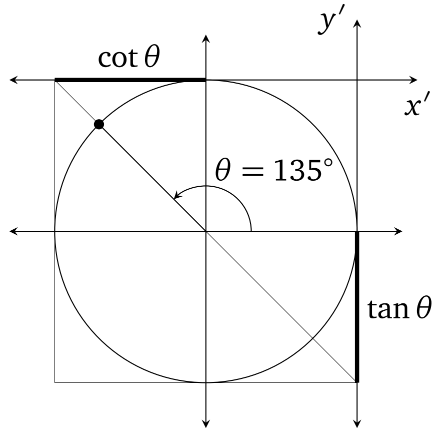
4.1 Exercise

Using the unit circle determine the tangent and cotangent of the following angles:

17. $0°$
18. $45°$
19. $90°$
20. $135°$
21. $180°$
22. $225°$
23. $270°$
24. $315°$
25. $360°$

Solutions:

17. $\tan 0° = 0$, $\cot 0°$ does *not* exist because the terminal side and the x' axis would never cross each other as they would be parallel!
18. $\tan 45° = 1$, $\cot 45° = 1$
19. $\tan 90°$ does *not* exist, $\cot 90° = 0$
20. $\tan 135° = -1$, $\cot 135° = -1$



21. $\tan 180° = 0$, $\cot 180°$ does *not* exist!
22. $\tan 225° = 1$, $\cot 225° = 1$
23. $\tan 270°$ does *not* exist, $\cot 270° = 0$
24. $\tan 315° = -1$, $\cot 315° = -1$
25. $\tan 360° = 0$, $\cot 360°$ does *not* exist!

4.2 Exercise

Using the unit circle, determine the sign of the tangent and cotangent of the following angles.

26. The terminal side of θ falls in the first quadrant.
27. The terminal side is in the second quadrant.
28. The terminal side is in the third quadrant.
29. The terminal side is in the fourth quadrant.

Solutions:

26. $\tan\theta > 0$, $\cot\theta > 0$
27. $\tan\theta < 0$, $\cot\theta < 0$
28. $\tan\theta > 0$, $\cot\theta > 0$
29. $\tan\theta < 0$, $\cot\theta < 0$

4.3 Exercise

Use the unit circle to answer the following questions.

30. What is the *maximum* possible value of the *tangent* function? At what value of θ this maximum occurs?
31. What is the *minimum* possible value of the *tangent* function? At what value of θ this minimum occurs?

32. What is the range of the *tangent* function?

    (What is the range of the values of the *tangent* function for any value of θ?)

33. What is the *maximum* possible value of the *cotangent* function? At what value of θ this maximum occurs?

34. What is the *minimum* possible value of the *cotangent* function? At what value of θ this minimum occurs?

35. What is the range of the *cotangent* function?

    (What is the range of the values of the *cotangent* function for any value of θ?)

Solutions:

30. The maximum value does *not* exist because as θ gets closer to 90°, $\tan\theta$ will get as large as you wish, in the positive or the negative direction!

31. Similarly, the minimum value does *not* exist because as θ approaches 270°, $\tan\theta$ will get arbitrarily large in the negative or positive direction!

32. All real numbers!

33. Similar to the tangent function, the maximum value does *not* exist because as θ approaches 0°, $\cot\theta$ will approach positive or negative infinity!

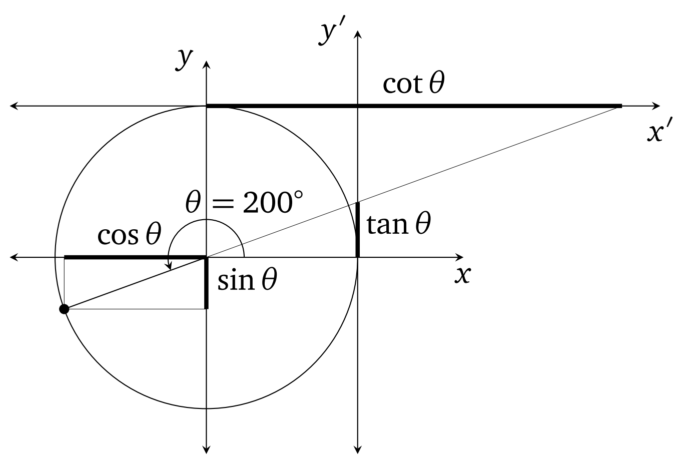
34. Similarly, the minimum value does *not* exist because as θ approaches 180°, $\tan\theta$ will get arbitrarily large in the negative or positive direction!

35. All real numbers!

4.4 Exercise

36. Draw the unit circle for $\theta = 200°$ and determine the line segments on the x' and y' axes that correspond to $\tan\theta$ and $\cot\theta$.

Solution:



Note that $\tan 200°$ and $\cot 200°$ are both *positive*.

5 Algebraic relations

The discussion so far was purely geometric. There are a few more trigonometric functions and many algebraic relations between them. Here we mention only a few algebraic relations and a couple of definitions.

The relations below can be geometrically verified on the unit circle, but we leave that as an exercise!

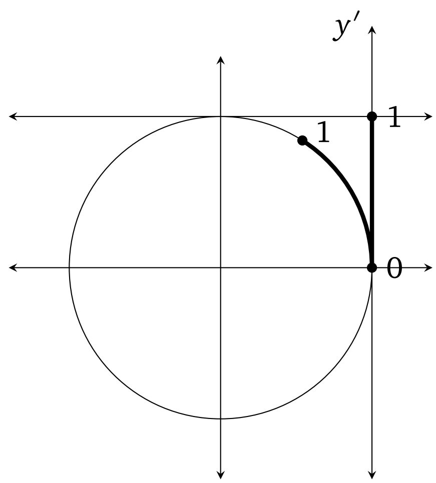
$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

The *secant* and *cosecant* functions are defined as the following. We skip their geometric representation.

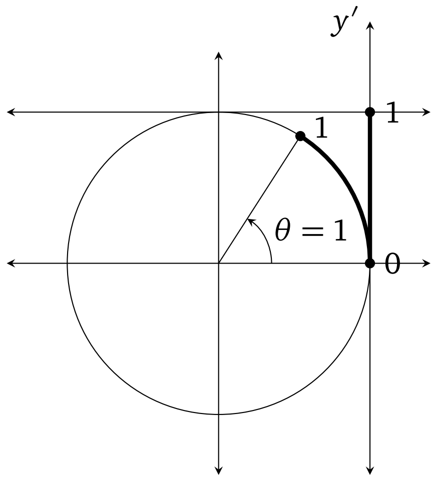
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

6 Radian: a unit of measurement of angles

- Let's say that the y' axis is made of some flexible elastic material that can be rolled around the unit circle!
  - We then proceed to roll the y' axis around the unit circle to find where the 1 tick mark lands of the circle!



  - The marked curve on the circle will then have a length of exactly 1!
  - The angle θ whose terminal side lands on the 1 tick mark on the unit circle is by definition an angle of size 1 *radian*—see the diagram below.
  - That is, we take the arc of length 1 in the unit circle as a unit of measurement of the central angle opposite to it, and we call it a radian.
  - So, *an angle of one radian is the angle centered at the origin and stretched under an arc of length 1 in the unit circle.*



- So a raidan equals an arc of unit length on the circumference of the unit circle.

6.1 Example

37. Knowing that a radian is equal to the unit length, how many radians is the circumference of the unit circle?

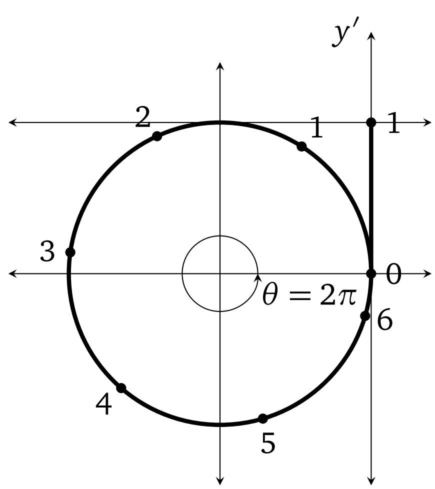
Solution:

It's been observed since antiquity that the ratio of the *circumference* C to the *diameter* D of circles of different size is the same. By definition, we denote this universal constant by the greek letter π.

$$\pi = \frac{C}{D} \qquad \text{(for any circle)}$$

Now we wonder what the circumference of the unit circle is. Well, since the diameter of the unit circle is 2, we have

$$C = 2\pi$$



7 Unit conversion for angles

- The example above shows that, *the angle that is stretched under a full circle is* 2π *radians*.
- But we know that the same angle measured in a different unit of measurement is 360°.
- Thus,
$$2\pi = 360^\circ$$
or
$$\boxed{\pi = 180^\circ} \tag{$*$}$$
- Using equation ($*$), we can convert degrees to radians and vice versa.

7.1 Example

If 180° is π radians, what is the measure of the following angles in raidans.

38. 30°
39. 45°
40. 60°
41. 90°
42. 180°
43. 270°
44. 360°

Solutions:

38. $30^\circ = 30^\circ \times 1 = 30^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{\pi}{6}$

    Note that $\dfrac{\pi}{180^\circ} = 1$.
39. Similarly $45^\circ = \frac{\pi}{4}$
40. $60^\circ = \frac{\pi}{3}$
41. $90^\circ = \frac{\pi}{2}$
43. $270^\circ = \frac{3\pi}{2}$